

A short proof of Stein's universal multiplier theorem

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Abstract

We give a short proof of Stein's universal multiplier theorem, purely by probabilistic methods, thus avoiding any use of harmonic analysis techniques (complex interpolation or transference methods).

1 Introduction

The celebrated Stein's universal multiplier theorem [Ste70, Corollary IV.6.3] provides strong (L^p, L^p) -bounds for a general family of operators related to a Markovian semigroup $(T^t)_{t \geq 0}$, virtually without any assumption on the underlying measure space (X, m) . More recent proofs of this classical result are based on analytic methods (see [CRW78, Cow81] and the monograph [CW76]), which also shows that the Markovianity assumption on the semigroup can be removed, keeping only the L^p -contractivity assumption. On the other hand, Stein's original proof relies on deep connections with martingale theory; not much later, P.A. Meyer began to investigate the problem purely by stochastic methods (see e.g. [Mey76] and subsequent articles, and also [Mey85] for an exposition of the transference approach).

In [Ste70], the multiplier theorem is actually a corollary of L^p -bounds for suitable Littlewood-Paley g -functions, which follows from a clever complex interpolation between the L^2 case, which holds by spectral theory, and an L^p -inequality, obtained by martingale tools. From a probabilist's viewpoint, this interpolation argument could be a mountain to climb: Meyer literally wrote that *on ne "comprend" pas ce qui se passe* [Mey76, end of Section 1].

In this note we prove the multiplier theorem (Theorem 1 below) relying only on martingale tools, namely Rota's construction and Burkholder-Gundy inequalities, the main contribution being therefore that we avoid the use of complex interpolation. With hindsight, this result could be considered as an analogue of the short proof of the maximal theorem for Markovian semigroups, sketched in [Ste70, below Theorem IV.4.9]: like in that case, powerful analytical tools can prove results for rather general semigroups but, in the Markovian setting, probability is enough and gives much simpler proofs. It is remarkable that, apparently, this shortcut went unnoticed, maybe because both Stein and Meyer were focusing mainly on Littlewood-Paley functions.

The proof allows also for an easy computation of the constants involved (in terms of p) and also for the norm of operators given by imaginary powers of the generator of the semigroup [Ste70, Corollary IV.6.4]. Assuming these bounds only, one can then deduce boundedness of g -functions [Med95, Theorem 1.1]. Another application (Corollary 4) comes from the fact that some form of Burkholder-Gundy inequalities still holds true for $p = 1$ (Davis' Theorem): we remark that one might also obtain analog results in a

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general, non-Markovian, setting by extrapolation on the L^p bounds (for a detailed account on Yano's extrapolation theory, see [KM05]).

After writing this note, we discovered that a similar argument already appeared in the last section of [Shi97]: there, however, continuous-time stochastic calculus is widely used and it is not fully recognized that Rota's construction and Burkholder-Gundy inequalities suffice, without any assumption on the underlying measure space.

2 Setting

We briefly recall the setting and notation of [Ste70, Chapters III and IV]. Let (X, dx) be a σ -finite measure space and let $(T^t)_{t \geq 0}$ be a strongly continuous semigroup of operators defined on $L^2(X, dx)$ such that the following conditions hold:

1. $\|T^t f\|_p \leq \|f\|_p$ ($1 \leq p \leq \infty$) (contraction);
2. T^t is self-adjoint on $L^2(X, \mu)$, for every $t \geq 0$ (symmetry);
3. $T^t f \geq 0$ if $f \geq 0$ (positivity);
4. $T^t 1 = 1$ (conservation of mass),

where $T^t 1$ is defined by $\sup_n T^t I_{A_n}$, taking a sequence $A_n \uparrow X$, with $I_{A_n} \in L^2$ (this is well defined in general because of positivity). The infinitesimal generator A of $(T^t)_{t \geq 0}$ in $L^2(X, dx)$ is given by

$$A = \lim_{t \downarrow 0} \frac{f - T^t f}{t},$$

for any $f \in L^2(X, \mu)$, whenever the limit exists in $L^2(X, dx)$ (that defines its domain $D(A)$).

Because of symmetry and contraction assumptions on T^t , the generator A is a self-adjoint, non-negative and densely defined operator. By spectral theory, there exists a unique resolution of the identity $(E(\lambda))_{\lambda \in \mathbb{R}}$ associated to A . In particular, the representation of $T^t = e^{-tA}$, $t \geq 0$, holds in the following sense:

$$\langle T^t f, g \rangle = \int_0^\infty e^{-t\lambda} d\langle E(\lambda) f, g \rangle, \quad (2.1)$$

where, for any $f, g \in L^2(X, dx)$, $\lambda \mapsto \langle E(\lambda) f, g \rangle$ is a bounded variation function on \mathbb{R} , with total variation not greater than $\|f\|_2 \|g\|_2$.

On the other side, positivity and conservation assumptions allows for a dynamical realization of the semigroup as the transition semigroup associated to a Markov process, with state space X and dx as invariant measure: this is the content of [Ste70, Theorem IV.4.9] (due to G.C. Rota), that we describe here in a more explicit form (and actually a bit simplified, as we require only a finite product space). Note that, having no assumption on X , it is not clear whether T^ε is induced by some probability kernel; still, the proof proceeds as in the case of existence of Markov chains.

Given $\varepsilon > 0$ and $N \in \mathbb{N}$, let $\Omega = X^{N+1}$, endowed with the product σ -algebra. For $k \in \{0, \dots, N\}$, let π_k be the projection on the k -th factor, let

$$\mathcal{F}_k = \sigma(\pi_k, \pi_{k+1}, \dots, \pi_N)$$

which defines an reverse (i.e. decreasing) filtration and let $\hat{\mathcal{F}} = \sigma(\pi_0)$. Then, there exists a σ -finite measure $\mathbb{P} = \mathbb{P}_{\varepsilon, n}$ on Ω such that the law of π_0 w.r.t. \mathbb{P} is dx and for $k \in \{0, \dots, N\}$, \mathbb{P} is σ -finite on \mathcal{F}_k and for every $f \in L^1(X, dx)$, it holds

$$T^\varepsilon f(\pi_0) = \hat{\mathbb{E}}[\mathbb{E}_k[f(\pi_0)]] = \hat{\mathbb{E}}[f_k],$$

where $\hat{\mathbb{E}}$ denotes the conditional expectation operator w.r.t. $\hat{\mathcal{F}}$, \mathbb{E}_k is the same, w.r.t. \mathcal{F}_k and $f_k = \mathbb{E}_k[f \circ \pi_0]$ is a reverse martingale: of course, being the set of times finite, there is no problem in applying the usual theory of martingales. Moreover, it is not difficult to check that all the properties and theorems used here and in what follows, which are well known to hold in probability spaces, extend verbatim to the σ -finite case, the only exception being Corollary 4 below.

We recall now the special case of spectral multipliers problem, addressed by Stein. Let M be a bounded Borel function on $(0, \infty)$ and define, for $\lambda > 0$,

$$m(\lambda) = -\lambda \int_0^\infty M(t) e^{-t\lambda} dt, \quad (2.2)$$

(let also $m(0) = 0$), which is a so-called multiplier of Laplace transform type, when we use it to define, by means of spectral calculus, the operator

$$T_m f = \int_0^\infty m(\lambda) dE(\lambda) f, \quad (2.3)$$

for $f \in L^2(X, dx)$. Since

$$\|m\|_\infty = \sup_\lambda |m(\lambda)| \leq \sup_t |M(t)| = \|M\|_\infty, \quad (2.4)$$

it follows by the spectral theorem that T_m is well defined and maps continuously $L^2(X, dx)$ into itself, with operator norm $\|T_m\|_{2,2} \leq \|M\|_\infty$. The problem consists in proving that, for $p \in]1, \infty[$, T_m maps continuously $L^2 \cap L^p(X, dx)$ into itself.

3 Proof of the multiplier theorem

We are in a position to state and prove Stein's result.

Theorem 1 (Stein's multiplier theorem). *Let $p \in]1, \infty[$. Then, T_m is a bounded linear operator on $L^2 \cap L^p(X, dx)$, with*

$$\|T_m\|_{p,p} \leq C_p \|M\|_\infty,$$

where $C_p = O((p-1)^{-1})$ as $p \downarrow 1$.

We sketch heuristically the line of reasoning. By substituting (2.2), which gives m in terms of M , into (2.3) and exchanging integrals, we obtain the expression

$$T_m f = \int_0^\infty M(t) \left[\int_0^\infty -\lambda e^{-t\lambda} dE(\lambda) f \right] dt = \int_0^\infty M(t) \frac{d}{dt} T^t f dt, \quad (3.1)$$

where we also used (2.1). Then, we formally simplify the increments dt and recall that, by Rota's construction, it holds $T^t f = \hat{\mathbb{E}}[f_t]$, where f_t is some reverse martingale:

$$T_m f = \int_0^\infty M(t) dT^t f = \hat{\mathbb{E}} \left[\int_0^\infty M(t) df_t \right].$$

To estimate the L^p norm, we use the fact that $\hat{\mathbb{E}}$ is a contraction and Burkholder-Gundy inequalities, obtaining

$$\|T_m f\|_p \leq C_p \|M\|_\infty \|f\|_p.$$

To make this reasoning rigorous, we first consider the case when M is a step function and then we pass to the limit. To do this, we state and prove two elementary lemmas, the first being in fact a special case of (3.1) for step functions.

Lemma 2. Given $N \in \mathbb{N}$, let $0 = t_0 \leq t_1 \leq \dots \leq t_N < \infty$ and let

$$M = \sum_{i=0}^{N-1} M_i I_{[t_i, t_{i+1}[} \cdot \quad (3.2)$$

Then, for every $f \in L^2(X, dx)$ it holds

$$T_m f = \sum_{i=0}^N M_i (T^{t_{i+1}} f - T^{t_i} f). \quad (3.3)$$

Proof. Since $M \mapsto m \mapsto T_m$ is linear, it is enough to consider the case $M = I_{[0, t]}$ and prove that $T_m = T^t - Id$. Integrating by parts, we have $m(\lambda) = e^{-t\lambda} - 1$ and so we conclude by (2.1). \square \square

Lemma 3. Let $(M^n)_{n \geq 0}$ be a sequence of Borel functions, with $\|M_n\|_\infty$ uniformly bounded and converging \mathcal{L}^1 -a.e. to some function M . Then, for every $f \in L^2(X, dx)$, it holds

$$\lim_{n \rightarrow \infty} T_n f = T_m f \quad \text{in } L^2(X, dx),$$

where T_n denotes the operator defined by M^n in place of M .

Proof. As above, by linearity, it is enough to consider the case $M = 0$ (i.e. $m = 0$ and $T_m = 0$). By dominated convergence, from (2.2) we obtain that, for every $\lambda \in [0, \infty[$, $|m_n(\lambda)|$ converges to zero. From (2.4) and the assumption on $(M^n)_{n \geq 0}$ this convergence is dominated by some constant, and this suffices to pass to the limit. Indeed, given $f \in L^2(X, dx)$, by spectral theorem, it holds

$$\|T_n f\|_2^2 = \langle T_n f, T_n f \rangle = \int_0^\infty |m_n(\lambda)|^2 d\langle E(\lambda) f, f \rangle.$$

As already remarked, $d\langle E(\lambda) f, f \rangle$ is a finite positive measure and so we conclude by dominated convergence. \square \square

Proof of Theorem 1. We may assume that $|M| \leq 1$. First, let M be a step function of the form (3.2), where $\varepsilon = t_{i+1} - t_i$ constant for $i \in \{0, \dots, N-1\}$. If we apply Rota's theorem, as described in the previous section, we obtain from Lemma 2 above that

$$T_m f \circ \pi_0 = \sum_{i=0}^{N-1} M_i \left(\hat{\mathbb{E}}[f_{i+1}] - \hat{\mathbb{E}}[f_i] \right) = \hat{\mathbb{E}} \left[\sum_{i=0}^{N-1} M_i (f_{i+1} - f_i) \right], \quad \mathbb{P}\text{-a.s. in } \Omega.$$

It holds therefore

$$\|T_m f\|_p = \left\| \hat{\mathbb{E}} \left[\sum_{i=0}^{N-1} M_i (f_{i+1} - f_i) \right] \right\|_p,$$

where the first norm is computed in $L^p(X, dx)$ and the other in $L^p(\Omega, \mathbb{P})$, because the law of π_0 is dx . Since conditional expectations are contractions, we have

$$\|T_m f\|_p \leq \left\| \sum_{i=0}^N M_i (f_{i+1} - f_i) \right\|_p.$$

We apply Burkholder-Gundy inequalities for martingale transforms (e.g. [Ste70, Theorem IV.4.2]) to the reverse martingale above:

$$\left\| \sum_{i=0}^N M_i (f_{i+1} - f_i) \right\|_p \leq c_p \|f\|_p,$$

where $c(p)$ is a constant, depending only on p : the claimed bound for $p \downarrow 1$ follows from constants-chasing in Marcinkiewicz interpolation. In the general case, we approximate a M with a sequence of step functions $(M^n)_{n \geq 1}$ such that $|M^n| \leq 1$ for every n and $M^n(s) \rightarrow M(s)$, \mathcal{L}^1 -a.e. $s \in [0, \infty[$: this possibility is well-known, as it follows e.g. by density in $L^1([0, \infty[, \mathcal{L}^1)$ of step functions and a diagonal argument. For a fixed $f \in L^2 \cap L^p(X, dx)$, Lemma 2 entails that, up to a subsequence, $(T_n f)$ converge dx -a.e. to $T_m f$. By Fatou's lemma, it holds

$$\|T_m f\|_p \leq \liminf_{n \rightarrow \infty} \|T_n f\|_p \leq c_p \|f\|_p,$$

that gives the thesis. \square

Corollary 4. *If (X, dx) has finite measure $|X|$, it holds for every $f \in L^2(X, dx)$,*

$$\|T_m f\|_1 \leq c |X| \|M\|_\infty \|f\|_{L \log L},$$

where $c > 0$ is some universal constant.

Proof. We may assume that $|X| = 1$ and $\|M\|_\infty \leq 1$. Arguing as above, we apply Davis' Theorem instead of Burkholder-Gundy inequalities, e.g. [LLP80, Theorem 2.1]:

$$\mathbb{E} \left[\left| \sum_{i=0}^N M_i (f_{i+1} - f_i) \right| \right] \leq c \mathbb{E} \left[\left(\sum_{i=0}^{N-1} |f_{i+1} - f_i|^2 \right)^{1/2} \right] \leq c \mathbb{E} \left[\sup_{i=0, \dots, N} |f_i| \right].$$

Then, we use the well-known corollary of Doob's inequality, that allows to control the L^1 norm of a maximal function of martingale (closed by f) in terms of the $L \log L$ norm of f . \square

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References

- [Cow81] M. G. Cowling, *On Littlewood-Paley-Stein theory*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), 1981.
- [CRW78] R. R. Coifman, R. Rochberg, and G. Weiss, *Applications of transference: the L^p version of von Neumann's inequality and the Littlewood-Paley-Stein theory*, Linear spaces and approximation, Birkhäuser, Basel, 1978.
- [CW76] R. R. Coifman and G. Weiss, *Transference methods in analysis*, American Mathematical Society, Providence, R.I., 1976.
- [KM05] G. E. Karadzhov and M. Milman, *Extrapolation theory: new results and applications*, J. Approx. Theory **133** (2005), no. 1, 38–99.
- [LLP80] E. Lenglart, D. Lépine, and M. Pratelli, *Présentation unifiée de certaines inégalités de la théorie des martingales*, Séminaire de Probabilités, XIV, Lecture Notes in Math., vol. 784, Springer, Berlin, 1980.
- [Med95] S. Meda, *On the Littlewood-Paley-Stein g -function*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2201–2212.
- [Mey76] P. A. Meyer, *Démonstration probabiliste de certaines inégalités de Littlewood-Paley. I. Les inégalités classiques*, Séminaire de Probabilités, X, Lecture Notes in Math., Vol. 511, Springer, Berlin, 1976, pp. 125–141.

- [Mey85] P.-A. Meyer, *Sur la théorie de Littlewood-Paley-Stein (d'après Coifman-Rochberg-Weiss et Cowling)*, Séminaire de probabilités, XIX, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 113–129.
- [Shi97] I. Shigekawa, *The Meyer inequality for the Ornstein-Uhlenbeck operator in L^1 and probabilistic proof of Stein's L^p multiplier theorem*, Trends in probability and related analysis (Taipei, 1996), World Sci. Publ., River Edge, NJ, 1997, pp. 273–288.
- [Ste70] E.M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J., 1970.